Inventory Control Using Fuzzy Dynamic Programming

Sadina Gagula-Palalic
and
Mehmet Can

Abstract
There are a variety of efficient approaches to solve crisp inventory models in operations research. In this article a model that uses Bellman and Zadeh's approach to fuzzy dynamic programming is used. The problem considered is the following: the management of a company wants to close down a certain plant within a definite time interval. Therefore production levels should decrease to zero as smoothly as possible and the stock level at the end of the planning period should be as low as possible. The demand is assumed to be deterministic.

Keywords: Fuzzy Dynamic Programming (FDA), Inventory control

1. Introduction

The earliest inventory control models were developed in the stochastic environment, such as economic order quantity model, which is applicable when the demand of an item has a constant or nearly constant rate. Japanese have built the method called just-in-time representing a philosophy whose objective is the elimination of all sources of waste including unnecessary inventory. The aim in any inventory model is to find the amount that should be ordered each period so that it would minimize the total cost, consisting of ordering and holding costs. However, like many other systems, inventory control includes the amount of uncertainty and as such can be modeled more efficiently by using Fuzzy logic modeling techniques. The first approach to fuzzy inventory control was introduced by Zadeh in 1965. In literature, various types of fuzzy inventory models were introduced and discussed by many researchers. For example, EOQ models with fuzzified parameters, such as demand, lead time and inventory level were presented by (Petrovic and Sweeney, 1994:147-152). Chen et al. (Chen-Wang, 1996:71-79) fuzzified the demand, ordering cost, inventory cost, and backorder cost into trapezoidal fuzzy numbers in EOQ model with backorder. Fuzzy multi-stage inventory problems were also considered by some researchers (e.g., Kacprzyk-Staniewski, 1982:117-32). This paper will present fuzzy dynamic programming techniques for modeling inventory, as a new and challenging approach. The drawbacks of fuzzy dynamic programming are that this is a method of solving problems exhibiting the properties of overlapping sub-problems that takes much less time than some naive methods.

2. Traditional Dynamic Programming

Traditional dynamic programming is a technique introduced first by Bellman in 1957. This technique is very known technique for solving large optimization problem that can be break up into small problems; once all the sub-problems have been solved, we are left with an optimal solution to the large problem. Each of the smaller problems is identified with a stage of the dynamic programming solution procedure. Basically the problem is formulated in terms of *state variables* x_n, representing the amount of inventory on hand at the beginning of stage n (n=1.2…N); *decision variables* d_n, representing the production quantity for stage n (n=1.2…N); *stage rewards*, r_n; a *reward function* $R_n(d_N, \dots, d_{N-n}, x_N)$; and a *transformation function* $t_n(d_n, x_n)$. The problem is solved by solving recursively the following:

$$\max_{d_n} R_n(x_n, d_n) = \max_{d_n} r_n(x_n, d_n)^{\circ} R_{n+1}(x_{n+1}) \qquad (1)$$

Such that

$$x_{n+1} = t_n(x_n, d_n) \qquad (2)$$
$$n = 1, 2, \dots, N-1$$

3. Fuzzy Dynamic Programming

Fuzzy dynamic programming was suggested first by Bellman and Zadeh in 1970. They based their considerations on the symmetrical model of a decision (Zimmermann, 2001:348):

Let

$\tilde{x}_n \in \tilde{X},\ n = 0, \dots, N$ - be defined as state variable where $\tilde{X} = \{\tau_1, \dots, \tau_N\}$ is the set of values permitted for the state variables;

$d_n \in \widetilde{D}, n = 0, \dots, N$ - be defined as decision variable where $\widetilde{D} = \{\propto_1, \dots, \propto_N\}$ is the set of possible decisions;

$x_{n+1} = t(x_n, d_n)$ - be the transformation function.

For stage $t, t = 1, \dots, N$, we define:

1. A fuzzy constraint $\tilde{C}_t$ limiting the decision space and characterized by its membership function $\mu_{\tilde{C}_t}(d_t)$
2. A fuzzy goal $\tilde{G}_N$ characterized by the membership function $\mu_{\tilde{G}_N}(x_N)$

The problem is to determine the maximizing decision $\widetilde{D}^0 = \{d_n^0\},\ n = 0, \dots, N$, for a given x_0.

A fuzzy set decision is the confluence of the constraints and goals and its membership function is defined by min-operator:

$$\widetilde{D} = \cap_{t=0}^{N-1} \tilde{C}_t \cap \tilde{G}_N \qquad (3)$$

$$\mu_{\widetilde{D}}(d_0, d_{N-1}) = \min\{\mu_{\tilde{C}_0}(d_0), \dots, \mu_{\tilde{C}_{N-1}}(d_{N-1}), \mu_{\tilde{G}_N}(x_N)\} \qquad (4)$$

The membership function of the maximizing decision is then

$$\mu_{\widetilde{D^0}}(d_0^0, \dots, d_{N-1}^0) = \max_{d_0, \dots, d_{N-2}} \max_{d_{N-1}} [\min\{\mu_{\tilde{C}_0}(d_0), \dots, \mu_{\tilde{C}_{N-1}}(d_{N-1}), \mu_{\tilde{G}_N}(t_N(x_{N-1}, d_{N-1}))\}] \qquad (5)$$

where d_n^0 represents the optimal decision on stage n.

4. Mathematical model for Fuzzy Dynamic Programming applied in Inventory Control

Let us assume that we have the following problem. A company needs to close down a certain plant within a definite time interval. The constraint is that the production level should be decreased as steadily as possible over this period. The goal is to provide that the stock levels are as low as possible at the end of the period. Therefore, the goal and constraints can be expressed as a fuzzy numbers, characterized by its membership function. In this case the demand is assumed to be crisp. The problem is set as follows:

Let
$d_n \in \widetilde{D}, n = 0, \ldots, N$
be the decision variable representing the production level in period n
$\widetilde{D} = \{\propto_1, \ldots, \propto_n\}$
is the set of the decisions allowed, a fuzzy set

$\tilde{x}_n \in \tilde{X},\ n = 0, \ldots, N$
be the state variable representing the stock level at the beginning of period n

$\tilde{X} = \{\tau_1, \ldots, \tau_m\}$
is the set of state possible value, a fuzzy set

$s_n = 1, \ldots, N$
is the crisp demand in period n

$x_{n+1} = x_n + d_n - s_n$
is the crisp transformation function

$\tilde{C}_n(d_n) = \{(d_n, \mu_{\tilde{C}_n}(d_n))\}$
are the fuzzy constraints representing "production should decrease as smoothly as possible"

$\tilde{G}_n(x_{N+1}) = \{x_{N+1}, \mu_{\tilde{G}}(x_{N+1})\}$ is the fuzzy goal representing "to have as low inventory level as possible"

In this case the demand is assumed to be crisp. However, objective functions as well as constraints can be non-crisp and therefore, they are defined by their membership functions. The aim is to maximize the goal within the ranges specified for the constraints according to mathematical expression (5).

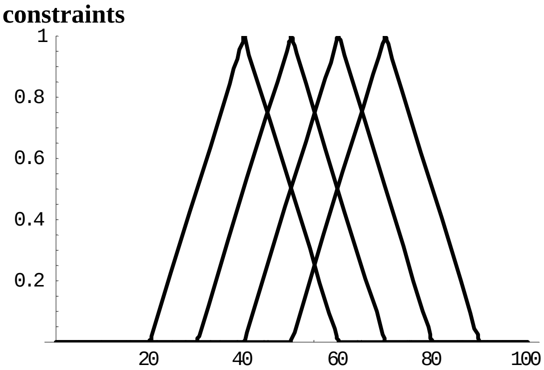
5. Application

Let the constraint be represented by the following membership function [Zimmerman(2001:428)]:

$$\mu_{C_n}(d_n) = \begin{cases} 0 & if \quad 0 \le d_n \le 60 - 10n \\ -3 + 0.5n + \dfrac{d_n}{20} & if \quad 60 - 10n \le d_n \le 80 - 10n \\ 5 - 0.5n - \dfrac{d_n}{20} & if \quad 80 - 10n \le d_n \le 100 - 10n \\ 0 & if \quad 100 - 10n \le d_n \end{cases}$$

For n=1,…,4, the membership functions are represented in the Figure 1.

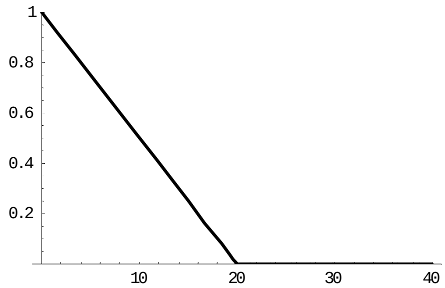
Figure 1- the membership functions of the constraints



Let the goal be represented by the following membership function:

$$\mu_{G_{N+1}}(x_{N+1}) = \begin{cases} 1 - x_{N+1}/20 & if \quad 0 \le x_{N+1} \le 20 \\ 0 & if \quad else \end{cases}$$

Figure 2 – the membership function of the goal



Let the number of stages be $N = 4$ and the non-crisp demands for each stage be

$$a_1 = 45, a_2 = 50, a_3 = 45, a_4 = 60$$

The sets of the values permitted for the decisions and state values are respectively:

$$D = \{0, 5, 10, \ldots\}, \;\; X = \{0, 5, 10, \ldots\}$$

Assume that the stock level at the beginning is $x_0 = 0$ and $0 < x_5 \leq 20$

Solution:

Since we are interested in the values of d_n for which $\mu_{C_n}(d_n) \geq 0$, the lower (l) and upper (u) bounds for the decision variables at different stages are found from the constraints membership functions as shown in the table 1.

Table1 – Lower and upper bounds for the decision variables

n	d_n^l	d_n^u
1	55	85
2	45	75
3	35	65
5	25	55

In order to find lower and upper bounds for the state variables, first forward calculation is performed and corresponding $x_{n,f}^l$ and $x_{n,f}^u$ are respectively found using the equations below and the results are shown in table 2.

$x_{n,f}^l = \max\{0, x_{n-1,f}^l + d_{n-1}^l - a_{n-1}\}$; $\quad x_{n,f}^u = x_{n-1,f}^u + d_{n-1}^u - a_{n-1}$; $\; n = 1, \ldots, 5$

Table 2 – Lower and upper bounds for the state variables by forward calculation

n	$x_{n,f}^l$	$x_{n,f}^u$
1	0	0
2	10	40
3	5	65
4	0	85

Second step is to perform backward calculation recursively and the following results are obtained:

Table 3 – Lower and upper bounds for the state variables by forward calculation

n	$x_{n,b}^l$	$x_{n,b}^u$
1	0	0
2	0	65
3	0	60
4	5	50
5	0	0

The final bounds for the state variables are obtained by the following equations and shown in the table 4:

$$x_n^l = max\{x_{n,f}^l, x_{n,b}^l\}$$
$$x_n^u = min\{x_{n,f}^u, x_{n,b}^u\}$$

Table 4 – Final lower and upper bounds for the state variables

n	x_n^l	x_n^u
1	0	0
2	10	40
3	5	60
4	5	50
5	0	0

At this point, it is possible to apply dynamic programming with fuzzy decision and state variables. The aim is to find the maximum value of the goal membership function for each state variable (whose range is given above in Table 4), and this is performed by applying equation (5).

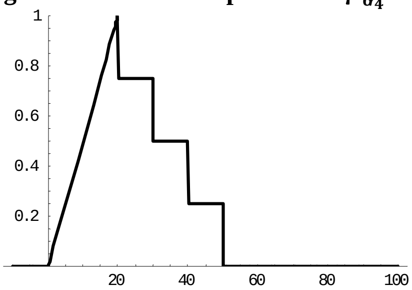
Stage 1 is obtained as follows: $\mu_{\tilde{G}_4}$is obtained from the following equation, results are shown in Table 5 and plotted in the Figure 3:

$$\mu_{\tilde{G}_4} = \max_{d_4}\{\min\,[\mu_C(d_4), \mu_G(x_4 + d_4 - a_4)\}$$

Table 5 – Stage 1

x_4	d_4: 25	30	35	40	45	50	55	$\mu_{G_4}(x_4)$
5	0	0	0	0	0	0	¼	¼
10	0	0	0	0	0	½	¼	½
15	0	0	0	0	¾	½	¼	¾
20	0	0	0	1	¾	½	¼	1
25	0	0	¾	¾	½	¼	0	¾
30	0	½	¾	½	¼	0	0	¾
35	¼	½	½	¼	0	0	0	½
40	¼	½	¼	0	0	0	0	½
45	¼	¼	0	0	0	0	0	¼
50	¼	0	0	0	0	0	0	¼

Figure 3 – Membership function $\mu_{\tilde{G}_4}$



Stage 2: The corresponding equation for the second stage is:

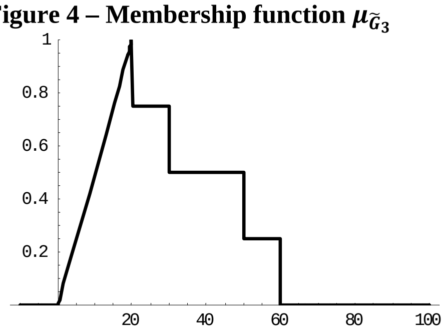
$$\mu_{\tilde{D}} = \max_{d_3}\{\min\,[\mu_C(d_3), \mu_G(x_3 + d_3 - a_3)\}$$

Function μ_G is obtained first by calculating the value of the transformation function $x_4 = x_3 + d_3 - a_3$, then taking the corresponding value from the abscissa vector of the function $\mu_{G_4}(x_4)$. The results are shown in the Table 6, and function $\mu_{G_3}(x_3)$ is shown in Figure 4.

Table 6 – Stage 2

x_3	d_3: 35	40	45	50	55	60	65	$\mu_{G_3}(x_3)$
5	0	0	¼	½	¾	½	¼	¾
10	0	¼	½	¾	¾	½	¼	¾
15	¼	½	¾	1	¾	½	¼	1
20	¼	½	¾	¾	¾	½	¼	¾
25	¼	½	¾	¾	½	½	¼	¾
30	¼	½	¾	½	½	¼	¼	¾
35	¼	½	½	½	¼	¼	0	½
40	¼	½	½	¼	¼	0	0	½
45	¼	½	¼	¼	0	0	0	½
50	¼	¼	¼	0	0	0	0	¼
55	¼	¼	0	0	0	0	0	¼
60	¼	0	0	0	0	0	0	¼

Figure 4 – Membership function $\mu_{\tilde{G}_3}$



Stage 3: Corresponding equation for decision membership function will be:

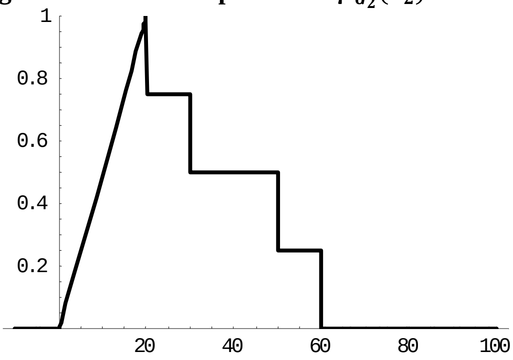
$$\mu_{\tilde{D}} = \max_{d_2}\{\min\,[\mu_C(d_2), \mu_G(x_2 + d_2 - a_2)\}$$

By using same procedure described in stage 2, the results for stage 3 are shown in the Table 7 and Figure 5.

Table 7 – Stage 3

x_2	d_2: 45	50	55	60	65	70	75	$\mu_{G_2}(x_2)$
10	¼	½	¾	1	¾	½	¼	1
15	¼	½	¾	¾	¾	½	¼	¾
20	¼	½	¾	¾	½	½	¼	¾
25	¼	½	¾	½	½	½	¼	¾
30	¼	½	½	½	½	½	¼	½
35	¼	½	½	½	½	¼	¼	½
40	¼	½	½	½	¼	¼	0	½

Figure 5 – membership function $\mu_{G_2}(x_2)$



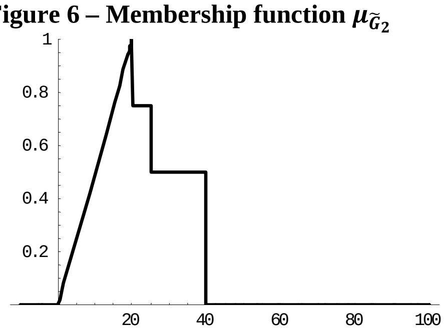
Stage 4: Corresponding equation for decision membership function will be:

$$\mu_{\tilde{D}} = \max_{d_1}\{\min\,[\mu_C(d_1), \mu_G(x_1 + d_1 - a_1)\}$$

Table 8 – Stage 4

	d_1							
x_1	55	60	65	70	75	80	85	$\mu_{G_1}(x_1)$
0	¼	½	¾	¾	½	½	¼	¾

Figure 6 – Membership function $\mu_{\tilde{G}_2}$



6. Conclusion

In this work, a new approach to inventory control is shown. The method described uses fuzzy dynamic programming, which has been proved as a powerful tool for optimization when non deterministic information exists. The complex problem can be subdivided into smaller problems and the state spaces were reduced by the introduction of a bound on the basis of heuristic considerations. Using a transformation function, upper and lower bounds for the state variables are found on the several intermediate stages and final solutions are found by fuzzy inference. Further work can be performed by introducing new fuzzy variables such as non crisp demand.

7. REFERENCES

Bellman R., Zadeh L.A., “Decision Making in Fuzzy Environment,” **Management Scie. 17, B-141-164.**

Chen S.H., Wang C.C. (1996:71-79), “Backorder Fuzzy inventory model under functional principle,” **Information Science.**

Kacprzyk, Staniewski (1981:117-32), “Long term inventory policy – making through fuzzy decision making models,” **Fuzzy Sets and Systems.**

Narasimhan R. (1980:325-338), “Goal Programming in a Fuzzy Environment,” **Decision Science**.

Petrovic D., Sweeny E. (1994: 147-152) (3), “Fuzzy Knowledge-based approach to treating uncertainty in inventory control,” **Computer Integrated Manufacturing Systems.**

Zimmerman H.J. (2001), “Fuzzy Set Theory and it’s Application,” **Kluwer.**