

## Optimization of Transport Problems with Fuzzy Coefficients

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### Abstract

In this paper, we concentrate on three kinds of fuzzy linear programming problems: linear programming problems with only fuzzy technological coefficients, linear programming problems with fuzzy right-hand sides and linear programming problems in which both the right-hand side and the technological coefficients are fuzzy numbers. We consider here only the case of fuzzy numbers with linear membership functions. The symmetric method of Bellman and Zadeh [2] is used for a defuzzification of these problems. The crisp problems obtained after the defuzzification are non-linear and even non-convex in general. Finally, we give illustrative examples and their numerical solutions.

Key Words: fuzzy set; fuzzy number; transport problems, fuzzy linear programming.

2000 Mathematical Subject Classification: 90C70, 90C26.

### 1. Introduction

In fuzzy decision making problems, the concept of maximizing decision was proposed by Bellman and Zadeh [2]. From the beginning of the theory, three periods in its development can be recognized:

#### 1. The period between 1965 - 1977.

This period is also known as the **academic phase**, which is characterized by the development of fundamentals of Fuzzy Set Theory, and speculations about possible prospective applications of the theory. The outcome was a small number of publications of a predominantly theoretical nature, by a small number of contributors, primarily from the academic community.

#### 2. The period between 1978 - 1988.

This period, also referred to as the **transformation phase**, is characterized by advances in Fuzzy Set Theory, but also by some practical applications of the theory. The number of contributors to the

theory also increased which resulted in increase of publications, some of which discussed various emerging applications. It is also important to say that some journals devoted to follow the development of this theory as well as to contribute to it were established.

#### 3. The period 1989 – nowadays.

This period, which is the current period, also referred to as the **fuzzy boom phase**, is characterized by a rapid increase in successful industrial and business applications of the theory which resulted in increase of revenues. Major research center devoted to applications of the theory were established, and brand new area of research, and if we may say, science emerged – *soft computing*. In the soft computing area, main partner of the Fuzzy Set Theory are *neural networks* and *genetic algorithms*.

The founder of the Fuzzy Set Theory is **Lotfi Asker Zadeh (Lotfi Aliasker Zadeh)**. He was born in *Baku, Azerbaijan, on February 4, 1921*. At the beginning of

his childhood he was raised in his family (mother Russian, father Azeri of Iranian origin) in his birth town, but he graduated high school in Teheran, at *Alborz High School (American Missionary School)* and got his *B.S. degree in Electrical Engineering at University of Teheran*. During Second World War Lotfi A. Zadeh moved to the United States, where he persuades his graduate studies and received *S.M. Degree in Electrical Engineering from the Massachusetts Institute of Technology in 1946*. Subsequently he joined the *faculty of Columbia University as an instructor of Electrical Engineering*, where he earned his *Ph.D. degree in 1949*.

Some of the most important publications of Lotfi A. Zadeh are the following:

1. Frequency analysis of variable networks, Proc. IRE 3a8, 291-299, 1950
2. A contribution to the theory of nonlinear systems, J. Franklin Institute 255, 387-408, 1953
3. Fuzzy sets, Inf. Control 8, 338-353, 1965
4. Trans. on Systems, Man and Cybernetics SMC-3, 28-44, 1973
5. Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1, 3-28, 1978

Some of the current statistical data show that this fresh-emerged theory has a big impact in scientific community.

A measure of the wide-ranging impact of Lotfi Zadeh's work on fuzzy logic is the number of papers in the literature which contain the word "fuzzy" in title. The data drawn from the INSPEC and Mathematical Reviews databases are summarized below. The data for 2000 are not complete.

1970-1980:	566
1980-1990:	2,361
1990-2000:	23,753
total:	26,680

#### **INSPEC/fuzzy 1**

1970-1980:	453
1980-1990:	2,476
1990-2000:	8,428
total:	11,357

#### **Math.Sci.Net/fuzzy 1**

Number of citations in the Citation Index: over 11,000.

The fuzzy logic concept was adapted to problems of mathematical programming by Tanaka et al. [13]. Zimmermann [14] presented a fuzzy approach to multiobjective linear programming problems. He also studied the duality relations in fuzzy linear programming. Fuzzy linear programming problem with fuzzy coefficients was formulated by Negoita [9] and called robust programming. Dubois and Prade [3] investigated linear fuzzy constraints. Tanaka and Asai [12] also proposed a formulation of fuzzy linear programming with fuzzy constraints and gave a method for its solution which bases on inequality relations between fuzzy numbers. Shaocheng [11] considered the fuzzy linear programming problem with fuzzy constraints and defuzzified it by first determining an upper bound for the objective function. Further he solved the so-obtained crisp problem by the fuzzy decisive set method introduced by Sakawa and Yana [10].

In this paper, we combine solution methods of Asai and Shaocheng, We first consider linear programming problems with fuzzy right-hand sides. Next we consider problems in which technological coefficients are fuzzy numbers and then finally linear programming problems in which both technological coefficients and right-hand-side numbers are fuzzy numbers. Each problem is first converted into an equivalent crisp problem. This is a problem of finding a point which satisfies the constraints and the goal with the maximum degree. The idea of this approach is due to Bellman and Zadeh [2]. The crisp problems, obtained by such a manner, can be non-linear (even non-convex), where the non-linearity arises in constraints. For solving these problems we use the computer algebra package MATHEMATICA.

The paper is outlined as follows. Linear programming problem with fuzzy technological coefficients is considered in Section 2.

Linear programming problem with fuzzy right-hand sides is considered in Section 3. In section 4, we study the linear programming problem with fuzzy objective function and fuzzy right hand sides. In section 5, several solution methods of the defuzzified problems are discussed. Section 6 is devoted to applications to concrete examples.

## 2. Linear programming problems with fuzzy right-hand side numbers

We consider a linear programming problem with fuzzy right-hand-side numbers

$$\begin{aligned} & \max \sum_{j=1}^n c_j x_j \\ & \text{subject to } \sum_{j=1}^n a_{ij} x_j < \tilde{b}_i, \quad i = 1, 2, \dots, m \quad (2.1) \\ & x_j \geq 0 \quad j = 1, 2, \dots, n \end{aligned}$$

where at least one  $x_j > 0$ . We make two assumptions.

Assumption 1.  $\tilde{b}_i$  is a fuzzy number with the following linear membership function:

$$B_i(x) = \begin{cases} 1 & \text{if } x < b_i \\ (b_i + p_i - x) / p_i & \text{if } b_i \leq x \leq b_i + p_i \\ 0 & \text{if } x > b_i + p_i \end{cases} \quad (2.2)$$

where  $x \in R$  and  $p_i > 0$  for all  $i = 1, 2, \dots, m$ . For defuzzification of this problem, we first fuzzify the objective function. This is done by calculating the lower and upper bounds of the optimal values first. The bounds of the optimal values,  $z_l$  and  $z_u$  are obtained by solving the following two standard linear programming problems

$$\begin{aligned} z_1 &= \max \sum_{j=1}^n c_j x_j \\ & \text{subject to } \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots, m \\ & x_j \geq 0 \quad j = 1, 2, \dots, n \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} z_2 &= \max \sum_{j=1}^n c_j x_j \\ & \text{subject to } \sum_{j=1}^n a_{ij} x_j \leq b_i + p_i, \quad i = 1, 2, \dots, m \\ & x_j \geq 0 \quad j = 1, 2, \dots, n \end{aligned} \quad (2.4)$$

The objective function takes values between  $z_1$  and  $z_2$  while right hand sides vary between  $b_i$  and  $b_i + p_i$ . Let  $z_l = \min(z_1, z_2)$  and  $z_u = \max(z_1, z_2)$ . Then,  $z_l$  and  $z_u$  are called the lower and upper bounds of the optimal values, respectively.

Assumption 2: The linear crisp problems (2.3) and (2.4) have finite optimal values.

In this case the fuzzy set of optimal values,  $\tilde{G}$  which is a subset of  $R^n$ ; is defined as (see Klir and Yuan [6]);

$$G(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{j=1}^n c_j x_j < z_l \\ \left( \sum_{j=1}^n c_j x_j - z_l \right) / (z_u - z_l) & \text{if } z_l \leq \sum_{j=1}^n c_j x_j \leq z_u \\ 0 & \text{if } \sum_{j=1}^n c_j x_j > z_u \end{cases} \quad (2.5)$$

The degree of satisfaction of the  $i$ th constraint by the  $n$  dimensional decision vector  $\mathbf{x}$  is given by

$$D_i(\mathbf{x}) = \begin{cases} 1 & \sum_{j=1}^n a_{ij}x_j \leq b_i \\ \left(b_i + p_i - \sum_{j=1}^n a_{ij}x_j\right) / p_i & b_i < \sum_{j=1}^n a_{ij}x_j < p_i + b_i \\ 0 & b_i + p_i \leq \sum_{j=1}^n a_{ij}x_j \end{cases} \quad (2.6)$$

By using the definition of the fuzzy decision proposed by Bellman and Zadeh [2] (see also Lai and Hwang [8]), we have

$$\tilde{D} = \left( \bigcap_{i=1}^m \tilde{D}_i \right) \cap \tilde{G} \quad (2.7)$$

In this case the optimal fuzzy decision is a solution of the problem  $\max \tilde{D}$ . Consequently, the problem (2.1) becomes to the following optimization problem

$$\begin{aligned} \max \quad & \lambda \\ G(\mathbf{x}) & \geq \lambda \\ D_i(\mathbf{x}) & \geq \lambda, \quad i = 1, 2, \dots, m \\ x_j & \geq 0, \quad j = 1, 2, \dots, n, \quad 0 \leq \lambda \leq 1. \end{aligned} \quad (2.8)$$

Using (2.5) and (2.6), the problem (3.8) can be written as a regular linear programming problem:

$$\begin{aligned} \max \quad & \lambda \\ \lambda(z_u - z_l) - \sum_{j=1}^n c_j x_j + z_u & \leq 0 \\ \lambda p_i - b_i - p_i + \sum_{j=1}^n a_{ij} x_j & \leq 0, \quad i = 1, 2, \dots, m \\ x_j & \geq 0, \quad j = 1, 2, \dots, n, \quad 0 \leq \lambda \leq 1. \end{aligned} \quad (2.9)$$

### 3. Solution of defuzzified problems

In the previous sections fuzzy decisive sets are obtained and the maximization problem of these sets is transformed into defuzzified problems. Notice that, the constraints in problems (2.9), (3.9) and (4.11) are generally not convex. These problems may be solved either by the fuzzy decisive set method, which is presented by Sakawa and

Yana [10], or by the linearization method of Kettani and Oral [6].

There are some disadvantages in using these methods. The fuzzy decisive set method takes a long time for solving the problem. On the other hand, the linearization method increases the number of the constraints.

Azimov and Yenilmez [5] presented a modified sub gradient method and used it for solving the defuzzified problems in fuzzy linear programming problems with linear membership functions. This method is based on the duality theory developed by Azimov and Gasimov [1] for nonconvex constrained problems and can be applied for solving a large class of such problems.

In the following applications to two fuzzy transport problems, we prefer to use Linear Programming Package in MATHEMATICA.

#### 4. Example: Solution of fuzzy transport problem by fuzzy linear programming

##### a. Fuzzy transport problem with fuzzy right-hand-side numbers

A company has two production centers  $F_1$  and  $F_2$ . Markets  $M_1$  and  $M_2$  sell the products of this company. The factory  $F_1$  produces daily at least 200 items. This production may go to 300, depending on the raw materials and the motivation of workers. Similarly the factory  $F_2$  produces at least 150 items a day and this production may go to 200. Market  $M_1$  sells 250 items a day, and it may increase it to 300 items. Market  $M_2$  sells 100 items a day, and it may increase it to 200 items.

The transportation costs from factories to market places are as shown in the following table:

<i>Market Place</i>	$M_1$	$M_2$
<i>Production Center</i>		
$F_1$	10	8
$F_2$	6	7

**Table 6. 1.**

The following table that shows the relation between production centers and market places will be helpful:

	$M_1$	$M_2$	Supply
$F_1$	$X_1$	$X_2$	$\tilde{F}_1$
$F_2$	$X_3$	$X_4$	$\tilde{F}_2$
Demand	$\tilde{M}_1$	$\tilde{M}_2$	$\tilde{C}_1 + \tilde{C}_2 = \tilde{F}_1 + \tilde{F}_2$

**Table 6. 2.**

Restrictions on demand and supply are not crisp. To make an ordering plan that will minimize the transportation cost minimum can be achieved via a fuzzy model. The objective function is

$$Z = 10x_1 + 8x_2 + 6x_3 + 7x_4 \quad (6.2)$$

Hence the fuzzy linear programming problem can be formulated as:

$$\min Z = 10x_1 + 8x_2 + 6x_3 + 7x_4$$

such that

$$x_1 + x_2 \prec \tilde{F}_1, x_3 + x_4 \prec \tilde{F}_2, \quad (6.3)$$

$$x_1 + x_3 \succ \tilde{M}_1, x_2 + x_4 \succ \tilde{M}_2,$$

$$x_1, x_2, x_3, x_4 \geq 0,$$

where membership functions for the above fuzzy sets and numbers can be defined as follows:

$$F_1(x) = \begin{cases} 1, & x \leq 200 \\ (300 - x)/100, & 200 < x < 300 \\ 0, & x \geq 300 \end{cases},$$

$$F_2(x) = \begin{cases} 1, & x \leq 150 \\ (200 - x)/50, & 150 < x < 200 \\ 0, & x \geq 200 \end{cases}, \quad (6.4)$$

$$M_1(x) = \begin{cases} 1, & x \leq 250 \\ (300 - x)/50, & 250 < x < 300 \\ 0, & x \geq 300 \end{cases},$$

$$M_2(x) = \begin{cases} 1, & x \leq 100 \\ (200 - x)/100, & 100 < x < 200 \\ 0, & x \geq 200 \end{cases},$$

For defuzzification of this problem, we first fuzzify the objective function. This is done by calculating the lower and upper bounds of the optimal values. The bounds of the optimal values,  $z_1$  and  $z_2$  are obtained by solving the following two standard linear programming problems

$$\min Z_1 = 10x_1 + 8x_2 + 6x_3 + 7x_4$$

such that

$$x_1 + x_2 \leq 200, x_3 + x_4 \leq 150, \quad (6.5)$$

$$x_1 + x_3 \geq 250, x_2 + x_4 \geq 100,$$

$$x_1, x_2, x_3, x_4 \geq 0,$$

and

$$\min Z_2 = 10x_1 + 8x_2 + 6x_3 + 7x_4$$

such that

$$x_1 + x_2 \leq 300, x_3 + x_4 \leq 200, \quad (6.6)$$

$$x_1 + x_3 \geq 300, x_2 + x_4 \geq 200,$$

$$x_1, x_2, x_3, x_4 \geq 0,$$

Linear Programming Module in Mathematica solves these two problems.

$$x_1 = 100, x_2 = 100, x_3 = 150, x_4 = 0,$$

$$Z_1 = Z_l = 2700$$

and

$$(6.7)$$

$$x_1 = 100, x_2 = 200, x_3 = 200, x_4 = 0,$$

$$Z_2 = Z_u = 3800.$$

If these results are used in (3.9), one gets  
 $\max \lambda$

$$\begin{aligned} (3800 - 2700)\lambda + (10x_1 + 8x_2 + 6x_3 + 7x_4) \\ - 3800 &\leq 0 \\ 100\lambda - 300 + x_1 + x_2 &\leq 0, \\ 50\lambda - 200 + x_3 + x_4 &\leq 0, \\ 50\lambda - 300 + x_1 + x_3 &\geq 0, \\ 100\lambda - 200 + x_2 + x_4 &\geq 0, \\ x_j \geq 0, \quad j = 1, 2, \dots, 4, \quad 0 \leq \lambda \leq 1. \end{aligned} \quad (6.8)$$

Linear Programming Module in Mathematica solves this problem:

$$\lambda = 0.5, \quad x_1 = 100, \quad x_2 = 150, \quad x_3 = 175, \quad x_4 = 0, \\ Z = 3250$$

Hence it is a compromise solution between  $Z_l$  and  $Z_u$ .

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