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Four-Color Coloring of a Partial Map of Europe

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Abstract

The four-color theorem states that any map in a plane can be colored using four-colors in such a way that regions sharing a common boundary other than a single point do not share the same color. In this article we attempt to color a partial map of Europe with four color using Artificial Intelligence techniques, defining it as a Constraint Satisfaction Problem (CSP). The algorithm created was succeeded to find all four solutions of the problem.

1. INTRODUCTION

Four Color Theorem has fascinated people for almost a century and a half. It dates back to 1852 when Francis Guthrie, while trying to color the map of counties of England, noticed that four colors suffice. He asked his brother Frederick if any map can be colored using four colors so that different colors appear on adjacent regions — that is, regions sharing a common boundary segment, not just a point. Frederick Guthrie then explained the problem to August DeMorgan, who in turn showed it to Arthur Cayley. The problem was first published as a puzzle for the public by Cayley in 1878.

The four-color theorem states that any map in a plane can be colored using four-colors in such a way that regions sharing a common boundary, other than a single point, do not share the same color. In mathematics, the four color map theorem, states that, given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color. Two regions are called adjacent if they share a common boundary that is not a corner, where corners are the points shared by three or more regions.

This problem is sometimes also called Guthrie's problem after F. Guthrie, who first conjectured the theorem in 1852. The conjecture was then communicated to de Morgan and thence into the general community. In 1878, Cayley wrote the first paper on the conjecture.
Three colors are adequate for simpler maps, but an additional fourth color is required for some maps, such as a map in which one region is surrounded by an odd number of other regions that touch each other in a cycle. The five color theorem, which has a short elementary proof, states that five colors suffice to color a map and was proven in the late 19th century (Heawood 1890, 1898); however, proving that four colors suffice turned out to be significantly harder.

The four color theorem was proven in 1976 by Kenneth Appel and Wolfgang Haken. It was the first major theorem to be proved using a computer. Appel and Haken's approach started by showing that there is a particular set of 1,936 maps, each of which cannot be part of a smallest-sized counterexample to the four color theorem (Appel, and Haken, 1977a). If they did appear, you could make a smaller counter-example. Appel and Haken used a special-purpose computer program to confirm that each of these maps had this property (Appel, and Haken, 1977b, 1986, 1989). Additionally, any map that could potentially be a counterexample must have a portion that looks like one of these 1,936 maps. Showing this required hundreds of pages of hand analysis. Appel and Haken concluded that no smallest counterexamples exist because any must contain, yet do not contain, one of these 1,936 maps. This contradiction means there are no counterexamples at all and that the theorem is therefore true. Initially, their proof was not accepted by all mathematicians because the computer-assisted proof was infeasible for a human to check by hand (Swart 1980). Since then the proof has gained wider acceptance, although doubts remain (Wilson 2004, 216–222).

To dispel remaining doubt about the Appel–Haken proof, a simpler proof using the same ideas and still relying on computers was published in 1997 by Robertson, Sanders, Seymour, and Thomas. Additionally, in 2005, the theorem was proven by Georges Gonthier with general purpose theorem proving software (Gonthier, 2008).

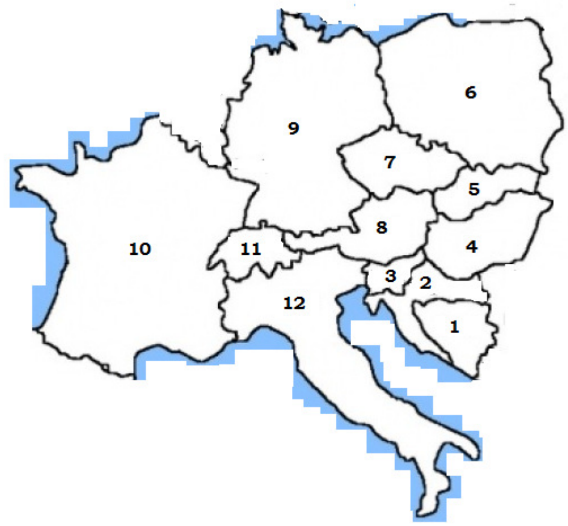
There are many great books on the 4-color problem. Here are three of the best, ordered from most accessible to most mathematical:

- Robin Wilson, Four Colors Suffice, Princeton University Press
- Rudolf Fritsch & Gerda Fritsch, The Four-Color Theorem, Springer
- Robert Wilson, Graphs, Colourlings and the Four-colour Theorem, Oxford Science Publications

2. METHOD

A partial map of European countries are as in the above. We are given the task of coloring each region either red, yellow, green, or blue in such a way that no neighboring countries have the same color.

To formulate this as a Constraint Satisfaction Problem, we define the variables to be the countries
X = {BH, CR, SN, HU, SK, PO ,CZ, AU, GE, FR, SW, IT}.



1. Bosnia and Hrz
2. Croatia
3. Slovenia
4. Hungary
5. Slovakia
6. Poland
7. Czech Republic
8. Austria
9. Germany
10. France
11. Switzerland
12. Italy

Figure 1. A Europe Partial Map, and numeration of 12 countries in it.

The domain of each variable is the set Di = {red, yellow, green, blue} . The constraints require neighboring countries to have distinct colors. Since there are 23 places where regions border, there are 23 constraints:

C ={(BH,CR), (CR, SN), (SN, IT), ...,(GE, PO)} .

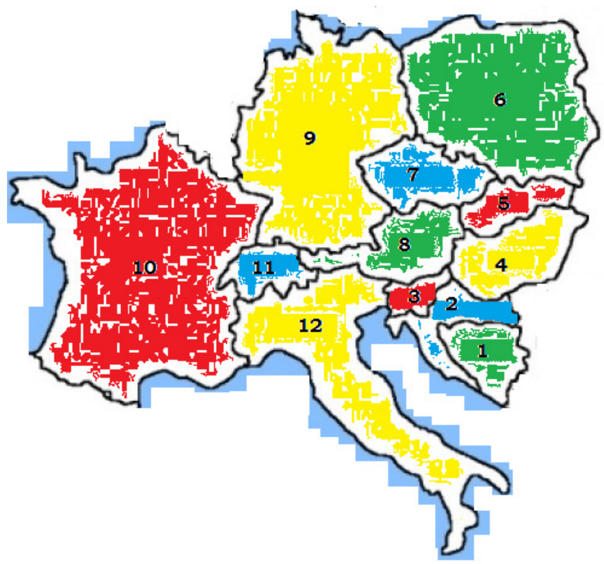


Figure 2. One version of coloring of Europe Partial Map in Figure 1.

To visualize this CSP, we plot the constraint graph. The nodes of the graph correspond to variables of the problem, and a link connects any two variables that participate in a constraint.

If we represent countries with their number labels:

X = {BH, CR, SN, HU, SK, PO, CZ, AU, GE, FR, SW, IT}
xn = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12};

the constraint graph for the problem becomes:

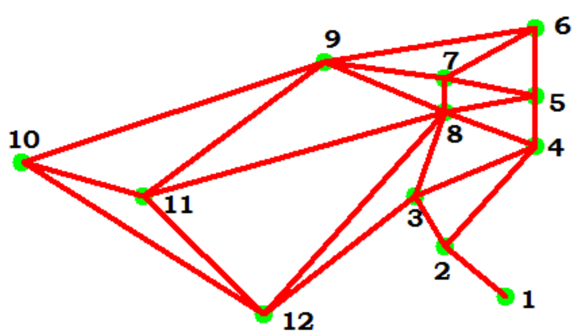


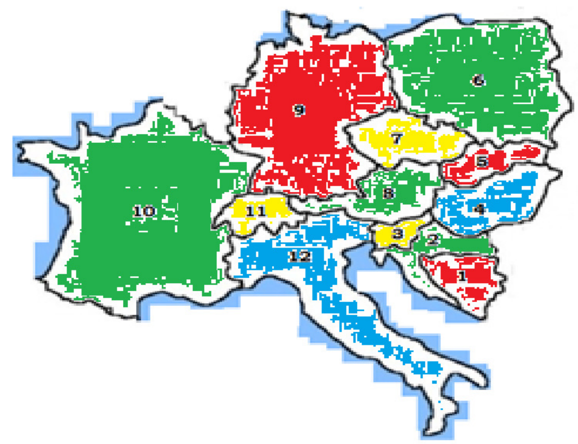
Figure 3. Constraint graph for the coloring problem of Europe Partial Map in Figure 1.

3. SOLUTION OF THE PROBLEM

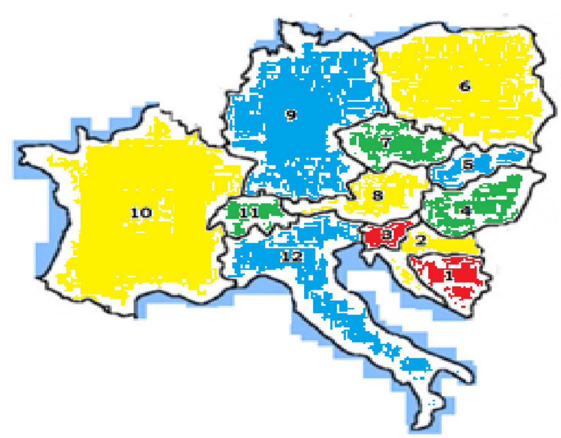
The complete state space consists of 12-tuples of four colors. Hence there are 16.777.216 incidences. A greedy algorithm searches this state space and finds 630 different solutions. Two of the solutions are as follows

Table 1. Two of the 630 different possible solutions

No	Country	C #1	C #2
1	Bosnia and Herzegovina	1	1
2	Croatia	2	3
3	Slovenia	3	1
4	Hungary	3	2
5	Slovakia	1	4
6	Poland	2	3
7	Czech Republic	3	2
8	Austria	2	3
9	Germany	1	4
10	France	2	3
11	Switzerland	3	2
12	Italy	4	4



(a)



(b)

Figure 4. Realization of two colorings in Table 1.

4. RESULTS AND DISCUSSION

It has been proved that in a map, if a country has odd number of neighbors more than one, then this map cannot be colored using only three colors. In partial Europe map, Austria has seven neighbors, therefore this map cannot be colored using only three colors. If we use the above coloring algorithm, the complete state space consists of 12-tuples of three colors, red, green, and blue. Hence there are 531.441 incidences. A greedy algorithm searches this state space and finds no solutions.

REFERENCES

Appel, K. and Haken, W. (1977a) Every Planar Map is Four-Colorable, II: Reducibility." Illinois J. Math. 21, 491-567, 1977.

Appel, K. and Haken, W. (1977b) The Solution of the Four-Color Map Problem." Sci. Amer. 237, 108-121.

Appel, K. and Haken, W. (1986) The Four Color Proof Suffices." Math. Intell. 8, 10-20 and 58.

Appel, K. and Haken, W. (1989) Every Planar Map is Four-Colorable. Providence, RI: Amer. Math. Soc.

Appel, K.; Haken, W.; and Koch, J. (1977b) Every Planar Map is Four Colorable. I: Discharging." Illinois J. Math. 21, 429-490.

Gonthier, G. (2008) Formal Proof—The Four-Color Theorem (PDF), Notices of the American Mathematical Society 55 (11), pp. 1382–1393.

Heawood, P. J. (1890) Map Colour Theorems, Quart. J. Math. 24, 332-338.

Heawood, P. J. . (1898) On the Four-Color Map Theorem." Quart. J. Pure Math. 29, 270-285.

Fritsch, F., and Fritsch, G. The Four-Color Theorem, Springer, 1998.

Swart, E. R. (1980), The philosophical implications of the four-color problem, American Mathematical Monthly (Mathematical Association of America) 87 (9), pp. 697–702.

Wilson, R. Four Colors Suffice : How the Map Problem Was Solved. Princeton, NJ: Princeton University Press, 2004.

Wilson, R. Graphs, Colourlings and the Four-colour Theorem, Oxford Science Publications (2002).